CONSERVATIVE MARKOV PROCESSES ON A TOPOLOGICAL SPACE

BY

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ABSTRACT

A Markov operator preserving $C(X)$ is known to induce a decomposition of the locally compact space X to conservative and dissipative parts. Two notions of ergodicity are defined and the existence of subprocesses is studied. A sufficient condition for the existence of a conservative subprocess is given, and then the process is assumed to be conservative. When it has no subprocesses, sufficient conditions for the existence of a σ-finite invariant measure are given, and are extended to continuous-time processes. When the invariant measure is unique, ratio limit theorems are proved for the discrete and continuous time processes. Examples show that some combinations of conservative processes are not necessarily conservative.

1. Definitions and notations. Let X be a locally compact perfectly normal space. We shall use the following properties of X:

(1.1) Every non-negative lower semi-continuous function is the limit-of an increasing sequence of non-negative continuous functions. [7, I(2)].

(1.2) If $\{V_\alpha\}_{\alpha\in I}$ is a collection of open sets, then the open set $V = \bigcup_{\alpha\in I} V_\alpha$ can be represented as the union of a countable subcollection $\{V_{\alpha_i}\}$. (Since $V = \bigcup B_n$ with B_n closed and σ-compact.)

(1.3) Every Borel set is a Baire set and consequently every Borel measure is regular [8, p.228] σ-finite (so every finite measure is regular).

We denote by Σ the collection of all Borel sets, by $B(X,\Sigma)$ the Banach space of all measurable bounded (real-valued) functions with the sup norm and by $C(X)$ the subspace of $B(X,\Sigma)$ consisting of continuous functions.

A *transition probability* on (X,Σ) is a function

$$P\colon X \times \Sigma \to [0,1]$$

satisfying:

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(1.4) $0 \leqq P(x,A) \leqq 1 \quad (x \in X, \quad A \in \Sigma)$

(1.5) $P(x, \ .)$ is a measure (countably additive) for each fixed $x \in X$.

(1.6) $P(. \ , A) \in B(X,\Sigma)$ for each fixed $A \in \Sigma$.

A transition probability P induces a positive contraction on $B(X, \Sigma)$, defined by:

(1.7) $Pf(x) = \int f(y)P(x,dy) \quad (f \in B(X, \Sigma))$.

P also induces a positive contraction on the space of finite signed measures $M(X,\Sigma)$, defined by:

(1.8) $\mu P(A) = \int P(x,A)\, \mu(dx) \quad (\mu \in M(X,\Sigma))$.

We shall frequently denote $\int f d\mu$ by $\langle \mu, f \rangle$, and we have

(1.9) $\langle \mu P,f \rangle = \langle \mu,Pf \rangle (\mu \in M(X,\Sigma), \quad f \in B(X,\Sigma))$.

In order to relate the transition probability to the topology, we assume

(1.10) $f \in C(X) \Rightarrow Pf \in C(X)$.

P will be called a *Markov process* on X if it satisfied (1.4)–(1.10).

LEMMA 1.1. *Let P be a Markov process on X; then:*

(a) *If $\{f_n\} \subseteq B(X,\Sigma)$ satisfies $\|f_n\| \leqq M$ for all n, and $f_n(x) \to f(x)$ for every $x \in X$, then $Pf_n(x) \to Pf(x)$ for every $x \in X$.*

(b) *If in* (a) *the convergence of $f_n(x)$ is non-decreasing, so is that of $Pf_n(x)$.*

(c) *If $0 \leqq f \in B(X,\Sigma)$ is lower semi-continuous, so is Pf.*

The proof is simple and will be omitted.

LEMMA 1.2. *If P is a Markov process on X, then:*

(a) *For every $0 \leqq f \in B(X,\Sigma)$ there is a minimal function f_∞ satisfying $Pf_\infty \leqq f_\infty$ and $f \leqq f_\infty \leqq \|f\|$.*

(b) *If $0 \leqq f \in B(X,\Sigma)$ is lower semi-continuous, so is f_∞.*

The proof of (a) is given in chapter III of [6]. (b) follows from the construction in [6] and lemma 1.1(c).

If $A \in \Sigma$ and $f = 1_A$ we shall denote f_∞ by i_A.

LEMMA 1.3. (a) *If $A_n \uparrow A$, then $i_{A_n} \uparrow i_A$.*

(b) *If $0 \leqq f \in B(X,\Sigma)$ satisfies $Pf \leqq f$, then the set $A = \{x: f(x) > 0\}$ satisfies $P1_A \leqq 1_A$.*

(c) *For every m, $P^m i_A \leqq \sum_{n=m}^{\infty} P^n 1_A$.*

PPROOF. (a) $\{i_{A_n}\}$ is increasing, so $g = \lim i_{A_n}$ exists. $Pg = P \lim i_{A_n} = \lim Pi_{A_n} \leqq \lim i_{A_n} = g$ so $Pg \leqq g$. $g \geqq \sup i_{A_n} \geqq 1_A$, so $i_{A_n} \leqq g$ by minimality. Since $i_{A_n} \leqq i_A$ for every n, $i_A \geqq g$.

(b) Define $A_n = \{x: f(x) > 1/n\}$. Then $1/n\, 1_{A_n} \leqq f$, so $i_{A_n} \leqq nf$, so $i_{A_n}(x) = 0$ for x outside A. Therefore, for such x, $i_A(x) = \lim i_{A_n}(x) = 0$ so $i_A = 1_A$.

(c) By minimality, $i_A \leqq \min\{1, \Sigma_{n=0}^{\infty} P^n 1_A\}$, so $P^m i_A \leqq \min\{1, \Sigma_{n=m}^{\infty} P^n 1_A\}$.
Q.E.D.

LEMMA 1.4. *If V is an open set and K is closed, then $V \cap K$ is nowhere dense if and only if it does not contain a non-empty open set.*

PROOF. $\overline{V \cap K} = \bar{V} \cap K$, and if it contains an open set $A \neq \phi$, then $\phi \neq A \cap V \subseteq V \cap K$. The converse is obvious. Note that by Baire's theorem a non-empty open set is of second category, so $V \cap K$ is either nowhere dense or of second category.

2. Conservative Markov Processes.

DEFINITION 2.1. An *inessential set* is a set $A \in \Sigma$ satisfying $\lim_{n\to\infty} P^n i_A(x) = 0$ for every $x \in X$. (Since $P i_A \leqq i_A$, $\{P^n i_A\}$ is a decreasing sequence, and the limit always exists.)

DEFINITION 2.2. The *dissipative part* of the process is the union of all inessential open sets and will be denoted by D. The *conservative part* of the process is the complementary set $C = X - D$. The process is *conservative* if $D = \phi$.

The following theorem was proved by Horowitz [9, theorem 2.2].

THEOREM 2.1. *There exists a representation $D = \bigcup_{n=1}^{\infty} D_n \cup N$, where N is a set of first category and each D_n is an open set satisfying $\Sigma_{k=0}^{\infty} P^k 1_{D_n} \in B(X, \Sigma)$.*

THEOREM 2.2. *If P is a Markov process on X, then the following conditions are equivalent:*

(a) *P is conservative.*

(b) *For every lower semi-continuous $0 \leqq g \in B(X, \Sigma)$ satisfying $Pg \leqq g$, the set $\{x: Pg(x) < g(x)\}$ is a set of the first category.*

(c) *For every lower semi-continuous $0 \leqq g \in B(X, \Sigma)$, the set $\{x: 0 < \Sigma_{n=0}^{\infty} P^n g(x) < \infty\}$ is of the first category.*

(d) *For every open set $U \neq \phi$, the set $U \cap \{x: \Sigma_{n=0}^{\infty} P^n 1_U(x) < \infty)$ is of the first category.*

(e) *If $0 \leqq g \in B(X, \Sigma)$ is lower semi-continuous and $P^n g \downarrow 0$, then $g \equiv 0$.*

PROOF. (a) ⇒ (b) [9, theorem 2.6].
(b) ⇒ (c) [5, theorem 9].
(c) ⇒ (d) is obvious.

(d) → (a): If $D \neq \phi$ then there is a non-empty D_n in the representation of the preceding theorem, since D is open and cannot be the first category. But $D_n \subseteq \{x: 0 < \sum_{k=0}^{\infty} P_k 1_{D_n}(x) < \infty\}$ which implies, by (d), that D_n is of the first category, a contradiction since D_n is open and non empty. Hence $D = \phi$.

(e) → (a) is immediate and (a) → (e) since $\{x: g(x) > 0\}$ is open, and by (b) of first category hence empty. Q.E.D.

THEOREM 2.3: *The following conditions are equivalent:*

(a) P *is conservative.*

(b) P^k *is conservative for every* k.

(c) P^k *is conservative for some* k.

PROOF.

(a) → (b): Let $0 \leqq g \in B(X,\Sigma)$ be a lower semi-continuous function satisfying $P^k g \leqq g$.

Define $f = (I + P + \cdots + P^{k-1})g$. Then $0 \leqq f \in B(X,\Sigma)$ is lower semi-continuous, and $f - Pf = g - P^k g \geqq 0$. Since P is conservative, we have by Theorem 2.2(b) that $\{x: Pf(x) < f(x)\} = \{x: P^k g(x) < g(x)\}$ is of the first category. Again by Theorem 2.2, P^k is conservative.

(b)⇒(c) is obvious, and (c)⇒(a) follows from Theorem 2.2(d), since for every open set U

$$U \cap \{x: \sum_{n=0}^{\infty} P^n 1_U(x) < \infty\} \subseteq U \cap \{x: \sum_{n=0}^{\infty} (P^k)^n 1_U(x) < \infty\}$$

Q.E.D.

REMARK. In the sequel, we give an example that the product of two commuting conservative Markov processes need not be conservative.

3. The conservative subprocess. We denote the complement for a set A by A'.

LEMMA 3.1. *If P is a Markov process on X and $Y \neq \phi$ is a closed subset satisfying $P 1_{Y'} \leqq 1_{Y'}$, then Q: Y $X(\Sigma \cap Y) \to [0,1]$, defined by $Q(y,A) = P(y,A)$ induces a Markov process on Y, and for every $f \in B(Y, \Sigma \cap Y)$ $Qf(y) = Pg(y)$ where g is any measurable extension of f to X.*

PROOF. Y clearly satisfies all our topological assumptions and Q is obviously a transition probability.

If g is any extension of f, then for $y \in Y$ $(P(y,Y') = 0)$:

$Pg(y) = \int g(z)P(y,dz) = \int_Y f(z)P(y,dz) = Qf(y)$.

Q satisfies (1.10): If $f \in C(Y)$, it can be extended to a $g \in C(X)$ (Tietze's theorem), and Pg is continuous, so Qf is continuous on Y. Q.E.D.

DEFINITION 3.1. A closed set Y with $P1_{Y'} \leqq 1_{Y'}$ is said to *define a subprocess.* The *subprocess* on Y is the above Q.

LEMMA 3.2. *A closed subset $Y \neq \phi$ defines a subprocess if and only if for every g and h in $C(X)$ coinciding on Y, $Pg = Ph$ on Y.*

PROOF. The condition is obviously necessary. Y' is open, and therefore $1_{Y'}$ is lower semi-continuous, and by (1.1) there is a sequence $\{f_n\} \subset C(X)$ with $0 \leqq f_n \leqq 1$ and $f_n \uparrow 1_{Y'}$. Since $f_n = 0$ on Y, $Pf_n = P0 = 0$ on Y, and by Lemma 1.1 for $y \in Y$

$$P\,1_{Y'}(y) = \lim Pf_n(y) = 0$$

or $P\,1_{Y'} \leqq 1_{Y'}$. Thus the condition is sufficient.

LEMMA 3.3. *If $\phi \neq A \subseteq X$, there is a minimal closed subset containing A and defining a subprocess.*

PROOF. We define

$$F = \{Z\colon\ A \subseteq Z,\ Z \text{ is closed},\ P1_{Z'} \leqq 1_{Z'}\}$$

F is not empty, since $X \in F$.

If $Y, Z \in F$, then $A \subseteq Y \cap Z$, and

$$P1_{(Y\cap Z)'} = P1_{Y'\cup Z'} \leqq P1_{Y'} + P1_{Z'} \leqq 1_{Y'} + 1_{Z'}.$$

If $x \in Y \cap Z$, $P1_{(Y\cap Z)'}(x) = 0$, therefore

$$P1_{(Y\cap Z)Y'} \leqq 1_{(Y\cap Z)'}, \text{ and hence } Y \cap Z \in F.$$

Define $B = \cap \{Z\colon Z \in F\}$. B is closed and contains A. By (1.2) B can be taken as the intersection of a sequence $\{Z_n\} \subseteq F$. As F is closed under finite intersections, we may take Z_n decreasing to B.

$$P1_{B'} = \lim P1_{Z_n'} \leqq \lim 1_{Z'_n} = 1_{B'}$$

by Lemma 1.1, so $B \in F$, and is minimal. Q.E.D.

In [5] and [9] it is proved that the conservative part C of the process defines a subprocess. (C is closed since D is open). It is not known if this subprocess is conservative in general, but it is if C is the closure of its interior, as a corollary of the following.

THEOREM 3.1. *Let C be the conservative part of the Markov process P on X. Then there is a decomposition $C = C_0 \cup C_1$, where C_0 is a nowhere dense set and C_1 is a closed set containing* int C *and defining a conservative subprocess.*

PROOF. If C has no interior put $C_0 = C$ and $C_1 = \phi$. Denote by V the interior of C and assume $V \neq \phi$. Let C_1 be the minimal closed subset containing V and defining a subprocess, which exists by Lemma 3.3, and is contained in C by minimality. $C_0 = C - C_1 \subseteq C - V$ is nowhere dense. It remains to show that the subprocess defined by C_1 is conservative. We shall prove first that V is contained in the conservative part of that subprocess. If this is not true then there is a relatively open set $A \subseteq C_1$ with $B = A \cap V \neq \phi$ and $P^n i_A \downarrow 0$ on C_1 (note that the minimal function on C_1 majorizing 1_A and subinvariant with respect to the subprocess is the restriction of i_A, defined in Lemma 1.2; this can be seen immediately from the construction [6, chapter III]). Since A is relatively open, $A = W \cap C_1$ with W open, and $B = A \cap V = V \cap W \cap C_1 = V \cap W$ is open, and satisfies $P^n i_B \downarrow 0$ on C_1, and especially $P^n i_B \downarrow 0$ on B. We now use a trick of Foguel: define (in X) $g = \lim P^n i_B$. By Lemma 1.1 $Pg = g$, and $g \geqq 0$. We got $g = 0$ on B, so $1_B \leqq i_B - g$, $P(i_B - g) \leqq i_B - g$ and by the minimality of i_B, $g \leqq 0$. Hence $g \equiv 0$ on X, and by definition 2.2 $B \subseteq D$, contradicting $\phi \neq B \subseteq C_1 \subseteq C$. Therefore V is contained in the conservative part C_2 of the subprocess defined by C_1. By minimality of C_1 (C_2 also defines a subprocess) $C_1 = C_2$. Q.E.D.

DEFINITION 3.1. *The conservative subprocess* of a Markov process P is the minimal subprocess containing the interior of the conservative part of P; it is conservative by the previous theorem.

EXAMPLE 3.1. $C_1 \neq \bar{V}$

$X = [0, 1] \cup \{2\}$ with the usual topology, $P(x,A) = \frac{1}{2}(1_A(1) + 1_A(2))$. A simple checking shows that $C = \{1, 2\}$. $\bar{V} = V = \{2\}$ but $C_1 = C$ since $\frac{1}{2} = P1_{V'} \nleqq 1_{V'}$.

EXAMPLE 3.2. $\phi \neq \bar{V} = C_1 \neq C$.

$X = [0,2]$ with the usual topology. Define $T(x) = \min\{x^2, x\}$ and $Pf(x) = f(T(x))$. $(1/n, 1)$ is open and inessential, and we can verify that $C = \{0\} \cup [1,2]$; but $\bar{V} = C_1 = [1,2]$, by the next theorem.

THEOREM 3.2. *If P is a Markov process on X with the property that $\{x\colon Pf(x) \neq 0\}$ is of the first category whenever $\{x\colon f(x) \neq 0\}$ is such a set ($f \in B(X, \Sigma)$), then the conservative subprocess is defined by $\bar{V}$* (the closure of the interior of C).

PROOF. We have to show that $\bar{V}$ defines a subprocess, and we shall use Lemma 3.2. Let $g, h \in C(X)$ with $g = h$ on $\bar{V}$. We define $\bar{g} = g1_C$ and $\bar{h} = h1_C$. $\{x: \bar{g}(x) \neq \bar{h}(x)\}$ is contained in $C - V$, which is of the first category. Hence $\{x: P\bar{g}(x) \neq P\bar{h}(x)\}$ is of the first category. Since $g = \bar{g}$ on C, $Pg = P\bar{g}$ on C (C defines a subprocess) and $Ph = P\bar{h}$ on C. Therefore, $\{x: Pg(x) \neq Ph(x)\} \cap C$ is of the first category, and cannot contain a non-empty open set by Lemma 1.4 (open sets $\neq \phi$ are of the second category by Baire's theorem), hence the open set $\{x: Pg(x) \neq Ph(x)\} \cap V = \phi$, and therefore $\{x: Pg(x) \neq Ph(x)\} \cap \bar{V} = \phi$, or $Pg = Ph$ on $\bar{V}$. The conclusion follows from Lemma 3.2. Q.E.D.

THEOREM 3.3. *Let P be a conservative Markov process on X. If Y is a closed subset defining a subprocess, it can be decomposed as $Y = A \cup B$, where A is nowhere dense and B a closed subset containing the interior of Y and defining a conservative subprocess.*

The proof is completely identical with that of Theorem 3.1, and will not be repeated.

THEOREM 3.4: *Let C be the conservative part of the Markov process on X, then:*

(a) *For every lower semi-continuous $0 \leqq g \in B(X, \Sigma)$ the set*

$$\{x: 0 < \Sigma P^n g(x) < \infty\} \cap C$$

is of first category.

(b) *For every lower semi-continuous $0 \leqq g \in B(X, \Sigma)$ with*

$$Pg \leqq g \text{ on } C, \{x: Pg(x) < g(x)\} \cap C$$

is of the first category.

PROOF. $C = C_0 \cup C_1$ by theorem 3.1: C_0 is of the first category, and on C_1 we have a conservative process to which we apply Theorem 2.2, noting that sets of first category in C_1 are such in X. Q.E.D.

4. Existence of subprocesses of a conservative process.

DEFINITION 4.1. A Markov process on X is *ergodic* if every non-empty closed set defining a subprocess is either equal to X or nowhere dense (has no interior).

THEOREM 4.1. *The following conditions are equivalent:*

(a) *P is conservative and ergodic.*

(b) *For every $0 \leqq g \in B(X, \Sigma)$ not identically zero lower semi-continuous function, $\{x: \Sigma_{n=0}^{\infty} P^n g(x) < \infty\}$ is of the first category.*

(c) *For every non-empty open set* U, $\{x\colon \sum_{n=0}^{\infty} P^n 1_U(x) < \infty\}$ *is of the first category.*

(d) *For every non-empty open set* U, $\{x\colon Pi_U(x) < 1\}$ *is of the first category.*

(e) *For every* $0 \leqq g \in B(X,\Sigma)$ *lower semi-continuous function satisfying* $Pg \leqq g$, $\{x\colon Pg(x) < \|g\|\}$ *is of the first category.*

PROOF. (a) $\Rightarrow$ (b): P is conservative, so $\{x\colon 0 < \sum_{n=0}^{\infty} P^n g(x) < \infty\}$ is of the first category by Theorem 2.2(c). Define $h = \min\{1, \sum_{n=0}^{\infty} P^n g\}$ then $Ph \leqq h$. $A = \{x\colon h(x) > 0\}$ satisfies (Lemma 1.3) $P1_A \leqq 1_A$ and is open since h is lower semi-continuous as soon as g is. Thus $X - A$ is a closed set defining a subprocess, and since $A \supseteq \{x : g(x) > 0\} \neq \phi$, $X - A$ has no interior by ergodicity, and $\{x\colon \sum_{n=0}^{\infty} P^n g(x) = 0\}$ is also of the first category.

(b) $\Rightarrow$ (c) is immediate.

(c) $\Rightarrow$ (d) P is conservative by Theorem 2.2(d).

By lemma 2.2 of [9] $\{x\colon i_U(x) > 0\} = \{x\colon \sum_{n=0}^{\infty} P^n 1_U(x) > 0\}$ and hence (c) implies $\{x\colon i_U(x) = 0\}$ is of the first category. $\{x\colon 0 < i_U(x) < 1\}$ is of the first category for P conservative, by theorem 2.4 of [9]. Hence $\{x\colon i_U(x) < 1\}$ is of first category. (d) follows by Theorem 2.2 (b).

(*d*) $\Rightarrow$ (e): For $g \equiv 0$ it is true, so assume $g \not\equiv 0$. $A_n = \{x : g(x) > \|g\| - \frac{1}{n}\}$ is not empty for large enough n.

$$1_{A_n} \leqq \frac{g}{\|g\| - 1/n}, \text{ and by minimality } i_{A_n} \leqq \frac{g}{\|g\| - 1/n}.$$

We have therefore

$$Pi_{A_n} \leqq (\|g\| - \frac{1}{n})^{-1} Pg, \quad \text{and} \quad \{x\colon Pg(x) < \|g\| - \frac{1}{n}\} \subseteq \{x\colon Pi_{A_n}(x) < 1\}$$

so the left-hand set is of first category by (d). Therefore $\{x\colon Pg(x) < \|g\|\}$ is of the first caregory as the union of a sequence of such sets.

(e) $\Rightarrow$ (a): Let $Y \neq X$ be a closed subset defining a subprocess.

Set $U = X - Y$, then $P1_U \leqq 1_U$, so $Y = \{x\colon 1_U(x) < 1\} \subseteq \{x\colon P1_U(x) < 1\}$ and both are of first category. Hence Y has no interior, and P is ergodic. Using Theorem 2.2(b) P is easily seen to be conservative if (e) holds. Q.E.D.

DEFINITION 4.2. A Markov process on X is *totally ergodic* if every non-empty closed set defining a subprocess is equal to X (there are no subprocesses).

LEMMA 4.1. *The following conditions are equivalent*:

(a) *P is totally ergodic.*

(b) *For every non-empty open set* U, $\{x: i_U(x) > 0\} = X$.

(c) *For every* $0 \leqq f \in C(X)$, *if* $f \not\equiv 0$ *then* $\sum_{n=0}^{\infty} P^n f(x) > 0$ *for every* x *in* X.

PROOF. By lemma 2.2 of [9], $\{x: i_U(x) > 0\} = \{x: \sum_{n=0}^{\infty} P^n 1_U(x) > 0\}$. If $U \neq \phi$ is open, there is a non-zero $f \in C(X)$ satisfying $0 \leqq f \leqq 1_U$, and hence (c) $\Rightarrow$ (b). If $0 \leqq f \in C(X)$ and $f \not\equiv 0$, $U = \{x: f(x) > a\}$ is not empty for some $a > 0$, and $a 1_U \leqq f$, so (b) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b): Let $U \neq \phi$ be open, and set $V = \{x: i_U(x) > 0\}$. V is open and by Lemma 1.3 $P1_V \leqq 1_V$, so $V' \neq X$ defines a subprocess, hence $V' = \phi$.

(b) $\Rightarrow$ (a): If Y is a closed set defining a subprocess, $P1_{Y'} \leqq 1_{Y'}$, or $1_{Y'} = i_{Y'}$ and by (b) Y' is ϕ or X; so P is totally ergodic. Q.ED

LEMMA 4.2. *In the following conditions,* (a) $\Rightarrow$ (b) $\Rightarrow$ (c).

(a) *For every non-empty open set* U, $Pi_U \equiv 1$.

(b) *For every non-empty open set* U, $\sum_{n=0}^{\infty} P^n 1_U \equiv \infty$.

(c) *P is conservative and totally ergodic.*

PROOF. (a) $\Rightarrow$ (b): The condition also implies $i_U \equiv 1$ and $P1 = 1$. Thus we have (by Lemma 1.3)

$$1 = P^m i_U(x) \leqq \sum_{n=m}^{\infty} P^n 1_U(x) \qquad (x \in X)$$

for every m, so the series diverges.

(b) $\Rightarrow$ (c): P is conservative by Theorem 2.2(d) and totally ergodic by the previous Lemma, since

$$\{x: i_U(x) > 0\} = \{x: \sum_{n=0}^{\infty} P^n 1_U(x) > 0\} = X$$

REMARKS.

1) If P is induced by a point transformation, i_U is always 0 or 1, so (c) $\Rightarrow$ $P1 = 1, i_U \neq 0 \Rightarrow Pi_U \equiv 1 \Rightarrow$ (a).

2) If there are no sets of first category (e.g. X countable with discrete topology, (c) $\Rightarrow$ (a).

3) S. Horowitz has shown the author a *probabilistic* proof that if X is compact then (c) $\Rightarrow$ (a).

4) It is not known if always (c) $\Rightarrow$ (b) or (b) $\Rightarrow$ (a).

5. Invariant (σ-finite) measures and ratio limits

DEFINITION 5.1. If μ is a Borel measure (positive, and finite on compact set σ-finite by σ-compactness of X), we define μP by means of (1.8). The measure μ is *subinvariant* if $\mu P \leqq \mu$, and *invariant* if equality holds.

For Markov processes defined on $L_1(X, \Sigma, \mu)$ our reference is [6].

LEMMA 5.1: *If P is a Markov process on X and μ is a subvariant Borel measure, then P defines a Markov process on $L_1(X, \Sigma, \mu)$.*

PROOF. $L_1(X, \Sigma, \mu)$ can be identified with a closed subspace of $M(X, \Sigma)$, and we have only to show that it is invariant under P. If $0 \leqq m$ is a measure weaker than μ, then $\mu(A) = 0$ implies $\langle \mu, P1_A \rangle = \mu P(A) = 0$, hence $\mu\{x: P1_A(x) > 0\}$ so $m\{x: P1_A(x) > 0\} = 0$ and $mP(A) = \langle m, P1_A \rangle = 0$. Q.E.D.

DEFINITION 5.2: We denote by $C_0(X)$ the linear manifold in $C(X)$ of continuous functions with compact support.

THEOREM 5.1. *Let P be a Markov process on X. If there is a function $0 \leqq g \in C_0(X)$ satisfying $\sum_{n=0}^{\infty} P^n g(x) = \infty$ for every $x \in X$, then there exists an invariant Borel measure.*

PROOF. Take any $0 \leqq f \in C_0(X)$. Since f has a compact support and $\sum_{n=0}^{\infty} P^n g \equiv \infty$, there is an integer K such that $\sum_{k=1}^{K} P^k g \geqq f$, and hence for every integer N $\sum_{n=0}^{N} P^n f \leqq \sum_{n=0}^{N} \sum_{k=1}^{K} P^{n+k} g \leqq K \sum_{n=0}^{N} P^n g + K^2 \|g\|$.

We now take a finite measure $m \neq 0$, and clearly $\sum_{n=0}^{\infty} \langle mP^n, g \rangle = \langle m, \sum_{n=0}^{\infty} P^n g \rangle = \infty$, so for $N \geqq N_0$ $\sum_{n=0}^{N} \langle mP^n, g \rangle > 0$.

We now have

$$\sum_{n=0}^{N} \langle mP^n, f \rangle \Big/ \sum_{n=0}^{N} \langle mP^n, g \rangle \leqq K + K^2 \|g\| \Big/ \sum_{n=0}^{N} \langle mP^n, g \rangle \to K.$$

Thus the sequence $\{ \sum_{n=n0}^{N} \langle mP^n, f \rangle \Big/ \sum_{n=0}^{N} \langle mP^n, g \rangle \}_{N=N_0}^{\infty}$ is bounded.

For any subsequence of integers N_j we define a linear functional ν on $C_0(X)$ by

$$\nu(f) = LIM \left\{ \frac{\sum_{n=0}^{N_j} \langle \Sigma mP^n, f \rangle}{\sum_{n=0}^{N_j} \langle mP^n, g \rangle} \right\} \quad \text{(A Banach limit)}$$

ν is a *positive* finite valued functional, so theorem D of [8, p. 247] applies to give a Borel measure μ such that $\nu(f) = \int f d\mu$ for $f \in C_0(X)$.

If $0 \leqq f \in C_0(X)$, then $0 \leqq Pf \in C(X)$, so we can find a sequence $\{f_n\} \subseteq C_0(X)$ with $0 \leqq f_n \uparrow Pf$.

If $0 \leqq h \in C_0(X)$ satisfies $h \leqq Pf$, then

$$\frac{\sum_{n=0}^{N_j}\langle mP^n, h\rangle}{\sum_{n=0}^{N_j}\langle mP^n, g\rangle} \leqq \frac{\sum_{n=0}^{N_j}\langle mP^n, Pf\rangle}{\sum_{n=0}^{N_j}\langle mP^n, g\rangle} \leqq \frac{\sum_{n=0}^{N_j}\langle mP^n, f\rangle}{\sum_{n=0}^{N_j}\langle mP^n, g\rangle} + \frac{m(X)\|f\|}{\sum_{n=0}^{N_j}\langle mP^n, g\rangle}$$

and letting $N_j \to \infty$ $\nu(h) \leqq \nu(f)$, as $\sum_{n=0}^{\infty}\langle mP^n, g\rangle = \infty$.

Hence

$$\langle \mu P, f\rangle = \int Pf d\mu = \lim \int f_n d\mu = \lim \nu(f_n) \leqq \nu(f) = \int f d\mu.$$

If B is a compact set, there is a decreasing sequence $\{h_n\}$ in $C_0(X)$ with $h_n \downarrow 1_B$ (by perfect normality).

$$\mu P(B) \leqq \lim_n \langle \mu P, h_n\rangle \leqq \lim_n \langle \mu, h_n\rangle = \mu(B).$$

Hence μP is finite on compact sets, and regular by (1.3), so $\mu P \leqq \mu$. By Lemma 5.1 P defines a process in $L_1(\mu)$. But $g \in C_0(X)$ implies $g \in L_1(\mu)$, and as $\sum_{n=0}^{\infty} P^n g(x) = \infty$ for every x, every x is in the conservative part of the adjoint process P^* [6, chapter VII]. But P and P^* on $L_1(\mu)$ have the same conservative part, so P on $L_1(\mu)$ is conservative, and the subinvariant measure μ is invariant by [6, chapter II]. Q.E.D.

THEOREM 5.2. *Let P be a Markov process on X, such that there is a function $0 \leqq g \in C_0(X)$ satisfuing $\sum_{n=0}^{\infty} P^n g(x) = \infty$ for every $x \in X$. If the invariant Borel measure μ is unique (up to a multiplicative constant), then for every finite measure m and every $f \in C_0(X)$ the following limit exists:*

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N}\langle mP^n, f\rangle}{\sum_{n=0}^{N}\langle mP^n, g\rangle} = \frac{\langle \mu, f\rangle}{\langle \mu, g\rangle}.$$

PROOF. It is enough to assume $f \geqq 0$. By the preceding theorem the sequence $\{a_N\}$ with

$$a_N = \sum_{n=0}^{N}\langle mP^n, f\rangle \Big/ \sum_{n=0}^{N}\langle mP^n, g\rangle$$

is bounded. If $\{a_{N_j}\}$ converges, we put N_j in the preceding theorem, so the limit equals $\nu(f)$ (ν is defined in the proof of the preceding theorem), which is, as is proved there, the integral of f with respect to an invariant measure giving mass 1 to g. Uniqueness of that measure implies that $\nu(f) = \langle \mu, f\rangle$. Thus $a_N \to \langle \mu, f\rangle / \langle \mu, g\rangle$.

Q.E.D

DEFINITION 5.3. *The kernel* of a Borel measure μ is the complement of the union of open sets on which μ vanishes.

In our topological set-up, we extend theorem 2 of [3].

THEOREM 5.3. *The kernel of a subinvariant Borel measure defines a subprocess.*

PROOF. Let μ be a subinvariant Borel measure (σ-finite by (1.3)) and K its kernel. Define $V = K'$, which is an open set, and is the union of a sequence of compact sets (by σ-compactness and perfect normality). Let $A \subseteq V$ be a compact set. We can find an $f \in C_0(X)$, satisfying $0 \leqq f \leqq 1$, $f(A) = 1, f(K(= 0$. Since μ is subinvariant $\int Pf \, d\mu = \int f \, d(\mu P) \leqq \int f \, d\mu = 0$. ($\mu(V) = 0$ as a consequence of (1.2) and the definition). Thus $Pf = 0$ a.e. Take $x \in K$. If $Pf(x) = a > 0$, then $\{y\colon Pf(y) > a/2\}$ is an open set with measure 0, so $x \in V$ — a contradiction. Hence for $x \in K$ $P1_A(x) \leqq Pf(x) = 0$. This being true for any $A \subseteq V$ compact, $P1_V(x) = 0$ for $x \in K$, or $P1_V \leqq 1_V$. By definition 3.1 K defines a subprocess. Q.E.D.

REMARKS. 1) The condition of Theorem 5.1 seems to be weaker than that of [4], but here we needed σ-compactness for the σ-finiteness of the invariant measure.

2) For the uniqueness requirement of Theorem 5.2, it is clearly necessary that the subprocess defined by the kernel of the invariant measure be *totally ergodic* (cf. S4) (otherwise the restriction of the invariant measure to a subprocess would define a different invariant measure).

THEOREM 5.4. *Let P be a Markov process on X. If for every $0 \leqq h \in C(X)$ and $h \not\equiv 0$ $\sum_{n=0}^{\infty} P^n h(x) = \infty$ for every $x \in X$ then there exists an invariant Borel measure μ. If it is unique, then for every $0 \leqq f$, $g \in C_0(X)$ and every finite measure m, the following limit exists:*

$$\lim_{N \to \infty} \frac{\sum_{n=0}^{N} \langle mP^n, f \rangle}{\sum_{n=0}^{N} \langle mP^n, g \rangle} = \frac{\langle \mu, f \rangle}{\langle \mu, g \rangle}$$

PROOF. The existence of μ follows from Theorem 5.1. The existence of the limit follows from applying Theorem 5.2 to each $0 \leqq g \not\equiv 0$ in $C_0(X)$.

REMARKS. 1) In Theorem 5.4 the assumption imply that P is conservative and totally ergodic (cf. Lemma 4.2).

2) The existence of an invariant measure under the conditions of theorem 5.4 was proved by Nelson [10, theorem 2.1]. The limit theorem in Theorem 5.4 was proved by Horowirtz [9] under the assumption $Pi_U \equiv 1$ for open sets $U \neq \phi$. His condition implies ours, and we do not know if they are equivalent (see Lemma 4.2 and remarks). However, his proof uses different techniques, which were shown by Foguel [7, §VI] to yield a result analogous to our Theorem 5.1 and 5.2.

For the uniqueness condition of Theorem 5.4, we can offer only the following criterion.

DEFINITION 5.4. A Markov process on X is *irreducible* if the measures λ_x, defined on Σ by $\lambda_x(A) = \sum_{n=1}^{\infty} 2^{-n}P^n(x,A)$, are all equivalent.

THEOREM 5.5. *If P is an irreducible Markov process on X, such that for every $0 \leqq h \in C(X)$ and $h \not\equiv 0$ $\sum_{n=0}^{\infty} P^n h \equiv \infty$, then it has a unique (up to a multiplicative constant) invariant Borel measure.*

PROOF. Let μ be an invariant measure. P in $L_1(\mu)$ is conservative by Theorem 5.1. The measures λ_x are absolutely continuous with respect to μ (theorem 8.1 of [1]) and if λ is equivalent to all λ_x, then P defines a process in $L_1(\lambda)$, which is therefore conservative too. By theorem 8.2 of [1] λ is necessarily equivalent to μ, and μ is unique. (P is also a *Harris* process, for which uniqueness is proved in [9, lemma 3.6]). Q.E.D.

REMARKS. 1) The example in [9] shows that irreducibility is *not* necessary.

2) The uniqueness assertion of the last theorem may be proved by showing that for every invariant measure μ, P on $L_1(\mu)$ is ergodic, as $\mu(A) > 0 \Rightarrow Pi_A > 0 \Rightarrow X$ is the only invariant set. The uniqueness now follows from the uniqueness of invariant measures for ergodic processes in L_1, by looking at μ_1 and $\mu_1 + \mu_2$ when μ_i are invariant. (The process is conservative in L_1.)

3) Irreducibility is more easily checked than the Harris condition of [9], as there is no need to know what the invariant measure is.

DEFINITION 5.5. A Markov process on X is *strongly conservative* if for every non-empty open set U, $\sum_{n=0}^{\infty} P^n 1_U(x) = \infty$ for every $x \in U$. Part (c) of the following lemma shows the motivation for this definition in analogy to processes on L_1.

LEMMA 5.2. *If P is a Markov process on X, then the following conditions are equivalent:*

(a) *P is strongly conservative.*

(b) *For every finite measure* m, $\sum_{n=0}^{\infty} mP^n(U) = \infty$ *for any open set* U *with* $m(U) > 0$.

(c) *For every finite measure* m *and open set* U, $\sum_{n=0}^{\infty} mP^n(U)$ *is either* 0 *or* ∞.

(d) *For any open set* U, $\sum_{n=0}^{\infty} P^n 1_U(x)$ *is either* 0 *or* ∞.

PROOF. (a) $\Rightarrow$ (b): For a finite measure m

$$\sum_{n=0}^{N} mP^n(U) = \left\langle \sum_{n=0}^{N} mP^n, 1_U \right\rangle = \left\langle m, \sum_{n=0}^{N} P^n 1_U \right\rangle$$

and if U is open with m$(U) > 0$, the right hand-side tends to ∞ as $N \to \infty$ since the integrand diverges on U, as P is strongly conservative.

(b) $\Rightarrow$ (c): If $\sum_{n=0}^{\infty} mP^n(U) \neq 0$, then for some k $mP^k(U) > 0$, and apply (b) to mP^k.

(c) $\Rightarrow$ (d) by inserting the Dirac measure δ_x as m.

(d) $\Rightarrow$ (a): For $x \in U$ and $U \neq \phi$ open, $\sum_{n=0}^{\infty} P^n 1_U(x) > 0$ so by (d) the sum is ∞. Q.E.D.

REMARKS. (1) A strongly conservative process is necessarily conservative, by Theorem 2.2(d).

(2) A conservative process may fail to be strongly conservative. Condition (d) of the last lemma is not satisfied in Example III (7) of [7].

LEMMA 5.3. *If* μ *is a subinvariant Borel measure for the strongly conservative Markov process* P, *then the process defined in* $L_1(X, \Sigma, \mu)$ *is conservative and* μ *is invariant.*

PROOF. The proof is similar to the end of the proof of Theorem 5.1 (we include every point x in a conditionally compact open set U with $\mu(U) > 0$ and get U in the conservative part).

LEMMA 5.4: *Let* P *be a strongly conservative Markov process on* X, *and let* U *be a conditionally compact open set. If there exists a finite measure* m *with* $m(U) > 0$ *such that for every conditionally compact open set* A

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} mP^n(A)}{\sum_{n=0}^{N} mP^n(U)} < \infty$$

then there exists an invariant Borel measure, which does not vanish on $\bar{U}$.

PROOF. A functional is defined on $C_0(X)$ as in the proof of Theorem 5.1, except that 1_U replaces g there. The proof is then identical for subinvariance, anσ invariance follows from Lemma 5.3.

COROLLARY: *The same holds if U is replaced by a compact set B.*

THEOREM 5.6. *If P is a strongly conservative Markov process on X, then the following condition is necessary and suffiicient for the existence of an invariant Borel measure: There exists a compact (or conditionally compact open) set B and a point $y \in B$, satisfying*

$$\limsup_{N\to\infty} \frac{\sum_{n=0}^{N} P^n 1_A(y)}{\sum_{n=0}^{N} P^n 1_B(y)} < \infty$$

for every conditionally compact open set A.

PROOF. The condition is sufficient by putting the Dirac measure δ_y in Lemma 5.4 or the corollary.

Necessity: Let μ be an invariant Borel measure. There is a compact (or conditionally compact open) set B with $0 < \mu(B) < \infty$, so $1_B \in L(X, \Sigma, \mu)$. By our topological assumptions $X = \cup A_i$ with A_i conditionally compact open sets. Using the Chacon-Ornstein theorem [6, chapter III] for P^* on $L_1(X, \Sigma, \mu)$ we have the existence a.e.(μ) on B of the finite limit

$$\lim_{N\to\infty} \frac{\sum_{n=0}^{N} P^n(x,A_i)}{\sum_{n=0}^{N} P^n(x,B)} < \infty \qquad (x \in B).$$

Therefore we can find a point $y \in B$ for which finite limits exist for all A_i's (the set of such y's in B has measure $\mu(B)$). If A is any conditionally compact open set, $\bar{A}$ (and hence A) can be covered by a finite number of A_i's, so the *lim sup* is bounded by a finite sum of finite limits. Q.E.D.

It is not known if a conservative totally ergodic process is necessarily strongly conservative (cf. Lemma 4.2 and remarks). If it is *not* true then the following lemma shows that P conservative need not imply invariance of a subinvariant Borel measure. (Compare with Lemma 5.3.)

LEMMA 5.5. *If P is a totally ergodic Markov process which is* not *strongly conservative then P has a subinvariant Borel measure which is* not *invariant.*

PROOF. Since P is not strongly conservative there is a $0 \leqq g \in C(X)$, $g \not\equiv 0$ and $\sum_{n=0}^{\infty} P^n g(x) < \infty$ for some $x \in X$.

By lemma 2.1 of [10], $\sum_{n=0}^{\infty} P^n f(x) < \infty$ for every $0 \leqq f \in C_0(X)$. Defining a linear functional on $C_0(X)$ by the sum, it defines a Borel measure μ which is easily seen to be subinvariant, and $\int Pf d\mu \leqq \sum_{n=1}^{\infty} P^n f(x) \leqq \int f d\mu$ for $0 \leqq f \in C_0(X)$ is proved in a way similar to the proof in Theorem 5.1. Clearly there is no equality when $f(x) > 0$. Q.E.D.

REMARK. This section treated σ-finite invariant measures. For the problem of finite invariant measures we refer to the appendix.

Professor Foguel has suggested to extend the results to the case of a continuous time process.

DEFINITION 5.6: A *continuous-time Markov process* is a family $\{P_t : 0 \leqq t < \infty\}$ of Markov processes such that the operators $\{P_t\}$ are a strongly continuous semigroup of operators on $C(X)$ (with $P_0 = I$). Some of the properties of a continuous-time process are described in [7].

THEOREM 5.7: *Let $\{P_t\}$ be a continuous-time Markov process. If there exists a function $0 \leqq g \in C_0(X)$ such that for every $x \in X$ $\int_0^{\infty} P_t g(x) dt = \infty$ then there exists a Borel measure μ satisfying $\mu P_t = \mu$ for every $t \geqq 0$.*

PROOF. Take $0 \leqq f \in C_0(X)$. For $S > 0$, $\int_0^S Pr\, g\, dr$ is continuous, so there is an S such that $\int_0^S P_r\, g\, dr \geqq f$, as $\int_0^{\infty} P_r\, g\, dr = \infty$ by hypothesis and f has compact support. Hence $0 \leqq \int_0^T P_t f\, dt \leqq \int_0^T (\int_0^S P_t P_r g dr) dt = \int_0^S (\int_0^T P_{t+r} g dt) dr = \int_0^S (\int_r^{T+r} P_t g\, dt) dr \leqq \int_0^S (\int_0^{T+S} P_t g dt) dr = S \int_0^T P_t g dt + S \int_T^{T+S} P_t g\, dt \leqq S \int_0^T P_t g dt + S^2 \|g\|$.

(The use of Fubini's theorem is justfied by the fact that the mapping $(t, r) \to P_{t+r} g(x)$, is continuous on $[0,\infty) \times [0,\infty)$). Let m be a finite measure on X. Then $0 \leqq \langle m, \int_0^T P_t f\, dt \rangle \leqq S \langle m, \int_0^T Ptg\, dt \rangle + S^2 \|g\| m(X)$ and since $\langle m, \int_0^T P_t g dt \rangle \to \infty$ as $T \to \infty$, we have

$$\limsup_{T \to \infty} \frac{\langle m, \int_0^T P_t f\, dt \rangle}{\langle m, \int_0^T P_t g\, dt \rangle} \leqq S$$

Furthermore for fixed r we have

$$\frac{\langle m, \int_0^T P_t P_r f\,dt\rangle}{\langle m, \int_0^T P_t g\,dt\rangle} \leqq \frac{\langle m, \int_0^T P_t f\,dt\rangle + \langle m, \int_T^{T+r} P_t f\,dt\rangle}{\langle m, \int_0^T P_t g\,dt\rangle} \leqq \frac{\langle m, \int_0^T P_t f\,dt\rangle + r\|f\|\, m(X)}{\langle m, \int_0^T P_t g\,dt\rangle}.$$

For any sequence $\{T_j\}$ increasing to ∞, we define a linear functional v on $C_0(X)$ by a Banach limit:

$$v(f) = LIM \left\{ \frac{\langle m, \int_0^{T_j} P_t f\,dt\rangle}{\langle m, \int_0^{T_j} P_t g\,dt\rangle} \right\} \quad f \in C_0(X).$$

v is well-defined as the sequence in the definition of $v(f)$ is bounded, and since v is positive, there exists a Borel measure μ such that $v(f) = \int f d\mu$ for $f \in C_0(X)$ [8, theorem D, p. 247]. By a similar argument to that of the proof of Theorem 5.1, and using the last inequality we have derived, we can conclude that $\mu P_r \leqq \mu$ for every $r \geqq 0$.

The function $(t,x) \to P_t g(x)$ is continuous on $[0,\infty) \times X$, as for any $\varepsilon > 0$

$$|P_t g(x) - P_r g(y)| \leqq |P_t g(x) - P_t g(y)| + \|P_t g - P_r g\| < \varepsilon$$

when y is in an appropriate neighborhood of x ($P_t g \in C(X)$) and r close enough to t.

By Lemma 5.1 each P_r defines a Markov process in $L_1(X, \Sigma, \mu)$, and $\langle \mu P_r, P_t g\rangle = \langle \mu, P_r P_t g\rangle$. Define $1P_r = d(\mu P_r)/d\mu$. We may use Fubini's theorem as $P_t g \geqq 0$ is continuous in both variables, so

$$\begin{aligned} 0 &\leqq \int_X (1 - 1P_r) \int_0^T P_t g\,dt\,d\mu = \int_X \int_0^T P_t g dt\,d\mu - \int_X 1P_r \int_0^T P_t g dt\,d\mu \\ &= \int_0^T \int_X P_t g d\mu dt - \int_0^T \int_X 1P_r \cdot P_t g d\mu dt \\ &= \int_0^T \langle \mu, P_t g\rangle\,dt - \int_0^T \langle \mu, P_r P_t g\rangle\,dt \leqq \int_0^r \langle \mu, P_t g\rangle\,dt \\ &= \int_0^r \langle \mu P_t, g\rangle\,dt \leqq \int_0^r \langle \mu, g\rangle\,dt \leqq r\,v(g) = r < \infty \end{aligned}$$

(all the integrals are finite valued, and bounded by $T\langle \mu, g\rangle = T$).
Letting $T \to \infty$, we conclude (for fixed r) $1 = 1P_r$ a.e. (μ), so $\mu P_r = \mu$. Q.E.D.

THEOREM 5.8. *Let $\{P_t\}$ be a continuous-time Markov process. If for every $0 \leqq h \in C(X)$ and $h \not\equiv 0$ and every $x \in X$ $\int_0^\infty P_t h(x)dt = \infty$, then there exists a Borel measure μ with $\mu P_t = \mu$ for all $t \geqq 0$. If it is unique,* *then for every* $0 \leqq f$, $g \in C_0(X)$ *and every finite measure m the following limit exists*:

$$\lim_{T\to\infty} \frac{\langle m, \int_0^T P_t f dt\rangle}{\langle m, \int_0^T P_t g dt\rangle} = \frac{\langle \mu, f\rangle}{\langle \mu, g\rangle}$$

PROOF. Completely analogous to that of Theorems 5.2, 5.4.

REMARK. The methods used in [9] did not extend to comntinuous-time process.

6. Strong ratio limit theorems. In Theorem 5.2 and 5.4, we obtained limit theorems involving the sums of the iterates of a process. In this section we look for a stronger ratio limit. In both of the following theorems we maintain the conditions of Theorem 5.2.

THEOREM 6.1. *Let P be a Markov process on X such that for some* $0 \leqq g \in C_0(X)$ $\sum_{n=0}^{\infty} P^n g(x) = \infty$ *for every* $x \in X$, *and assume that the invariant measure* μ *is unique. If m is a finite measure on X satisfying* $\limsup_{n\to\infty} \langle mP^{n+1}, f\rangle / \langle mP^n, f\rangle \leqq 1$ *for every* $0 \leqq f \in C_0(X)$ *with* $f \not\equiv 0$, *then for every* $0 \leqq f \in C_0(X)$ *and integer r the following limit exists*:

$$\lim_{n\to\infty} \langle mP^{n+r}, f\rangle / \langle mP^n, g\rangle = \langle \mu, f\rangle / \langle \mu, g\rangle$$

PROOF. Fix f. As $\sum_{n=0}^{\infty} P^n g = \infty$, if $0 \leqq h \in C_0(X)$, then there exists an integer J such that $\sum_{j=0}^{J} P^j g \geqq h$, because h has a compact support. If $\delta > 0$, then the condition imposed on m yields $\langle mP^{n+j}, g\rangle / \langle mP^n, g\rangle \leqq (1+\delta)^j$ for $n \geqq N$, and

$$\frac{\langle mP^n, h\rangle}{\langle mP^n, g\rangle} \leqq \frac{\sum_{j=0}^{J} \langle mP^{n+j}, g\rangle}{\langle mP^n, g\rangle} \leqq \sum_{j=0}^{J} (1+\delta).$$

Thus the sequence $\langle mP^n, h\langle / \langle mP^n, g\rangle$ is bounded for $n \geqq N$ and every $h \in C_0(X)$.

Let $\{n_i\}$ be a subsequence such that $\langle mP^{n_i}, f\rangle / \langle mP^{n_i}, g\rangle$ converges. We define a positive linear functional on $C_0(X)$ by a Banach limit:

$$v(h) = LIM \left\{ \frac{\langle mP^{n_i}, h\rangle}{\langle mP^{n_i}, g} \right\} \qquad h \in C_0(X).$$

We apply theorem D of [8, p.247] and get a Borel measure λ such that $v(h) = \int h d\lambda$.

Fix $0 \leqq h \in C_0(X)$. For any $\varepsilon > 0$, we have hypothesis for n_i large enough:

$$\frac{\langle mP^{n_i+1},h\rangle}{\langle mP^{n_i},g\rangle} = \frac{\langle mP^{n_i},h\rangle}{\langle mP^{n_i},g\rangle}\,\frac{\langle mP^{n_i+1},h\rangle}{\langle mP^{n_i},h\rangle} \leqq (1+\varepsilon)\,\frac{\langle mP^{n_i},h\rangle}{\langle mP^{n_i},g\rangle}.$$

Since $0 \leqq P\,h \in C(X)$, there is a sequence $\{h_k\}$ in $C_0(X)$ with $h_k \uparrow Ph$. Thus

$$\int Phd\lambda = \lim_k \int h_k d\lambda = \lim_k \nu(h_k) \leqq LIM \left\{\frac{\langle mP^{n_i+1},h\rangle}{\langle mP^{n+i},g\rangle}\right\} \leqq$$
$$\nu(h)(1+\varepsilon) = (1+\varepsilon)\int hd\lambda.$$

Letting $\varepsilon \to 0$ we get $\int Phd\lambda \leqq \int hd\lambda$ for every $0 \leqq h \in C_0(X)$, and λ is therefore a subvariant measure, and hence invariant as is proved at the end of the proof of Theorem 5.1. By the uniqueness of the invariant measure $\lambda = \alpha\mu$, and as $\nu(g) = 1$, $\nu(h) = \langle\mu,h\rangle/\langle\mu,g\rangle$, and hence

$$\lim\langle mP^{n_i},f\rangle/\langle mP^{n_i},g\rangle = \nu(f) = \langle\mu,f\rangle/\langle\mu,g\rangle.$$

As this is true for every convergent subsequence, the theorem is proved for $r = 0$.

We next show that $\langle mP^{n+1},g\rangle/\langle mP^n,g\rangle \to 1$. If $\{n_i\}$ is a subsequence for which $\langle mP^{n_i+1},g\rangle/\langle mP^{n_i},g\rangle$ converges, we put this subsequence in the definition of ν, and putting $h = g$ in the equality $\int Phd\lambda = \int hd\lambda$, we have that

$$\langle mP^{n_i+1},g\rangle/\langle mP^{n_i},g\rangle \to \nu(g) = 1.$$

$$\lim\frac{\langle mP^{n+r},f\rangle}{\langle mP^n,g\rangle} = \lim\frac{\langle mP^{n+r},f\rangle}{\langle mP^{n+r},g\rangle}\prod_{i=1}^{r}\frac{\langle mP^{n+i},g\rangle}{\langle mP^{n+i-1},g\rangle} = \frac{\langle\mu,f\rangle}{\langle\mu,g\rangle}$$

and the theorem is proved.

THEOREM 6.2. *Let P be a Markov process on X such that there is a function $0 \leqq g \in C_0(X)$ satisfying $\sum_{n=0}^{\infty}P^n g(x) = \infty$ for every $x \in X$, and assume that the invariant measure μ is unique. If m is a finite measure satisfying*

$$\liminf_{n\to\infty}\frac{\langle mP^n,f\rangle - \langle mP^{n+1},f\rangle}{\langle mP^n,g\rangle} \geqq 0 \text{ for } 0 \leqq f \in C_0(X)$$

then for every $f \in C_0(X)$ and every integer r the following limit exists:

$$\lim_{n\to\infty}\langle mP^{n+r},f\rangle/\langle mP^n,g\rangle = \langle\mu,f\rangle/\langle\mu,g\rangle$$

PROOF. Putting $f = g$ in the given condition, we obtain $\limsup\langle mP^{n+1},g\rangle/\langle mP^n,g\rangle \leqq 1$ and since $\sum_{n=0}^{\infty}P^n g \equiv \infty$ implies $\sum_{n=0}^{\infty}\langle mP^n,g\rangle = \infty$, equality holds.

If $0 \leqq h \in C_0(X)$, then the sequence $\{mP^n, h\rangle / \langle mP^n, g\rangle\}$ is bounded (for $n \geqq N$), as is proved at the beginning of the preceding theorem.

Fix $0 \leqq f \in C_0(X)$. If $\{n_i\}$ is a subsequence for which $\{mP^{n_i},f\rangle / \langle mP^{n_i} g\rangle\}$ converges, we define a positive linear functional on $C_0(X)$ by a Banach limit:

$$\nu(h) = LIM\left\{\frac{\langle mP^{n_i},h\rangle}{\langle mP^{n_i}g\rangle}\right\} \quad h \in C_0(X).$$

By [8,p.247, theorem D) there exists a Borel measure λ satisfying $\nu(h) = \int hd\lambda$.

Take $0 \leqq h \in C_0(X)$. $0 \leqq Ph \in C(X)$ and there exists a sequence $\{h_k\}$ in $C_0(X)$ with $0 \leqq h_k \uparrow Ph$. For any $\varepsilon > 0$ the given condition implies that for $n \geqq n_0(h.\varepsilon)$ we have $\langle mP^{n+1},h\rangle / \langle mP^n,g\rangle \leqq \langle mP^n,h\rangle / \langle mP^n,g\rangle + \varepsilon$ and hence

$$\int Phd\lambda = \lim_k \int h_k d\lambda = \lim_k \nu(h_k) \leqq LIM\,\{\langle mP^{n_i+1},h\rangle / \langle mP^{n_i},g\rangle\} \leqq \nu(h) + \varepsilon =$$
$$= \int hd\lambda + \varepsilon.$$

Letting $\varepsilon \to 0$ we have $\int Phd\lambda \leqq \int hd\lambda$ for every $0 \leqq h \in C_0(X)$, and, as in the preceding theorem, λ is an invariant measure, and therefore a constant multiple of μ. As $\nu(g) = 1$ we have

$$\lim_{n_i \to \infty} \langle mP^{n_i},f\rangle / \langle mP^{n_i},g\rangle = \nu(f) = \langle \mu,f\rangle / \langle \mu,g\rangle .$$

The end of the proof goes word by word as that of the preceding theorem. Q.E.D.

REMARKS. 1) It is obvious that the condition imposed on m in Theorem 6.2 is necessary.

2) Foguel [7, chapter VII] proved a theorem similar to our Theorem 6.2. His assumptions are of a local nature and so is his result (concerning continuous functions supported in a given compact set). Our method of proof is of a global nature and so are our results but unfortunately so are our conditions (concerning all $C_0(X)$).

3) Under the assumptions of Theorem 5.4 the condition of Theorem 6.1 becomes $\lim\sup \langle mP^{n+1},f\rangle / \langle mP^n,f\rangle = 1$ $0 \leqq f \in C_0(X)$. The theorem can be applied to each $0 \leqq g \in C_0(X)$, and the condition is necessary, as $\langle \mu,f\rangle > 0$ for $0 \leqq f \in C_0(X)$, $f \not\equiv 0$.

4) For an ergodic and conservative chain with n-step transition probabilities $p_{ij}^{(n)}$ our result is that if (for fixed i) $\lim\sup\, p_{ij}^{(n+1)}/p_{ij}^{(n)} = 1$ for every j, then $p_{ij}^{(n+1)}/p_{ik}^{(n)} \to \mu_j/\mu_k$ for every j,k and r. This is still asking for much more than Orey's condition [11].

7. Combinations of conservative processes. It is well known that the product of two Markov processes is again a Markov process, and so is a convex linear combination of two Markov processes. If the two processes are conservative, what can be said about these new processes? The following examples show they they are *not* necessarily conservative, even when the original processes commute.

EXAMPLE 7.1. *A product of strongly conservative processes is not conservative.* Let Z be the set of all integers. We define $X = \{(x,y,z):x,y,z \in Z\}$ with the discrete topology. We define 3 processes: P – a symmetric random walk parallel to the x-axis, Q parallel to the y-axis and R parallel to the z-axis. P, Q, R are conservative. The probability of returning to the origin after 2^n steps is $a_{2n} = \binom{2n}{n} 2^{-2n}$ in each of the processes. P, Q, R all commute. We look at $P(QR)$. QR is isometric to the two dimensional random walk (going to 4 points with probability $\frac{1}{4}$), and is conservative [2, chapter XIV.7]. By independence, the probability v_{2n} of return to the origin by PQR after 2^n steps is $[\binom{2n}{n} 2^{-2n}]^3$. $\sum_{n=0}^{\infty} v_{2n} < \infty$, since $v_{2n} \sim \Pi^{-3/2} n^{-3/2}$ by approximation with Stirling's formula, and thus $P(QR)$ is not conservative.

EXAMPLE 7.2. *A convex linear combination of strongly conservative processes is not conservative.* We define X, P, Q, R as above, and consider $\frac{1}{3}R + \frac{2}{3}PQ$. This process is isomorphic to the symmetric 3-dimensional random walk, which is known not to be conservative [2, chapter XIV.7].

REMARKS: 1) In our examples the counting measure was invariant for all processes.

2) These examples apply also to processes in $L_1(\mu)$ — take for μ the counting measure. In that case the process is even Harris process [6, chapter V].

Appendix — Existence of a finite invariant measure. This problem will be treated in [7]. We give only the following result.

THEOREM. *Let P be a Markov process on X. If A is a compact set, then the following conditions are equivalent* (*we assume* $P1 = 1$):

(a) $\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \not\equiv 0$.

(b) $\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \not\equiv 0$.

(c) $\limsup_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} P^n 1_A \right\| > 0$.

(d) *There exists a finite invariant measure* μ *with* $\mu(A) > 0$.

(e) *There exists a finite measure* m *satisfying* $\liminf mP^n(A) > 0$.

(f) *There exists a finite measure* m *satisfying*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} mP^n(A) > 0.$$

(g) *There exists a finite measure* m *satisfying*

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} mP^n(A) > 0.$$

PROOF. (a) ⇒ (b) ⇒ (c) is obvious.

(c)⇒(d) can be deducted from [3]. A more detailed proof will appear in [7, §IV].

(d) ⇒ (a) by the ergodic theorem, since

$\lim \frac{1}{N}\sum_{n=1}^{N} P^n 1_A(x)$ exists a.e. μ and the limit g satisfies $\int g d\mu = \int 1_A d\mu = \mu(A) > 0$.

(d) ⇒ (e) with $m = \mu$ and clearly (e) ⇒ (f) ⇒ (g)

(g) ⇒ (c) since

$$\frac{1}{N}\sum_{n=1}^{N} mP^n(A) = \left\langle m, \frac{1}{N}\sum_{n=1}^{N} P^n 1_A \right\rangle \leqq m(X) \left\| \frac{1}{N}\sum_{n=1}^{N} P_A \right\|$$

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